THE HYPERGEOMETRIC FUNCTIONS OF N VARIABLES*

BY

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The generalization of the hypergeometric series in one variable and three parameters to that in two variables and four parameters was first made by APPELL in 1880.† He showed that this series satisfies a system of partial differential equations of the second order, and that any four solutions of these equations are connected by a linear relation. Immediately after, PICARD ‡ showed that on the assumption of three linearly independent integrals which behave in a prescribed way in the vicinity of certain singularities, the differential equations of APPELL were completely deducible. APPELL also expressed the hypergeometric series in terms of a double definite integral in which the two variables occur as parameters, while PICARD expressed it in a more useful form as a simple definite integral. These results were generalized by LAURICELLA.§

The next question to be considered was the behavior of the three linearly independent solutions of the hypergeometric differential equations when the two independent variables described closed paths about the singular points. It was found by PICARD || that they undergo linear transformations, any of which can be expressed by a combination of five particular ones. PICARD also showed that a certain Hermitian form is invariant for the substitutions of the group, but he went no further than to establish this fact for the cases in which the four parameters in the hypergeometric functions are rational numbers.

In the following paper I consider the hypergeometric functions of n variables $x_1, x_2, \cdots, x_n$ and $n+2$ parameters. I deduce the generating substitutions of the group of linear homogeneous transformations on the $n+1$ linearly independent hypergeometric integrals produced by a variation of the x_i along closed

* Presented to the Society April 25, 1908.

† His results were published in a completed form in a memoir entitled *Sur les fonctions hypergéometriques de deux variables*, Journal de Mathematiques, ser. 3, vol. 8 (1882), pp. 173–216.

‡ *Sur une extension aux fonctions de deux variables*, etc., Journal de l'École Normale, ser. 2, vol. 10 (1881), pp. 305–322.

§ *Sulle funzioni ipergeometriche a piu variabili*, Rendiconti del Circolo Matematico di Palermo, vol. 7 (1893), pp. 111–158.

|| *Sur les groupes de certaines équations différentielles linéaires*, Bulletin des Sciences Mathématiques, vol. 9 (first part, 1885), pp. 202–209; *Sur les fonctions hyperfuchsiennes*, etc., Journal de l'École Normale, ser. 2, vol. 14 (1885), pp. 357–384.

paths surrounding the critical points. I then show that a certain Hermitian form is invariant for this group.

§ 1. *The group of linear transformations.*

Denote by U the function

$$(u-u_0)^{\alpha-1}(u-u_1)^{\beta-1}(u-x_1)^{\lambda_1-1}(u-x_2)^{\lambda_2-1}\cdots(u-x_n)^{\lambda_n-1}$$

and regard the $n+1$ hypergeometric integrals

$$\omega_0=\int_{u_0}^{x_1} Udu, \qquad \omega_i=\int_{u_1}^{x_i} Udu \qquad\qquad (i=1, 2, \cdots, n)$$

as functions of the variables x_i. In order that these integrals may have a meaning it is necessary that the real parts of α, β, λ_i, and $(n+1)-(\alpha+\beta+\lambda_1+\lambda_2+\cdots+\lambda_n)$ be positive. For convenience we will assume $u_0=0$, $u_1=1$ as long as there is no restriction of generality. The following abbreviations will also be used:

$$a=e^{2\pi i\alpha}, \qquad b=e^{2\pi i\beta}, \qquad l_k=e^{2\pi i\lambda_k}.$$

Suppose that x_1 describes a closed path positively about the zero point. By expanding ω_0 in ascending powers of x_1 it is easily seen to be reproduced multiplied by al_1 when x_1 undergoes this change. The path of integration for ω_1 is altered into the one indicated by the dotted lines in Fig. 1. The integral taken along the portion of the path from 1 to 0 is equal to $\omega_1-\omega_0$. In going around the point 0 the factor $u^{\alpha-1}$ in the integrand is reproduced multiplied by a. This factor occurs in every element of the integral (regarded as the limit of a sum) along the path $0x_1$ and we obtain $a\omega_0$ for the result of integration along the second part of the path. Hence, denoting the transformed integral by ω_1', we have

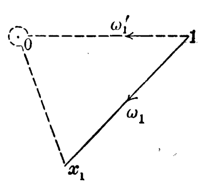

FIG. 1.

$$\omega_1'=(a-1)\omega_0+\omega_1.$$

The remaining integrals $\omega_i(i>1)$ are not affected by this monodromy of the branch-point x_1. Hence we obtain the substitution

$$(\Sigma_0) \qquad \omega_0'=al_1\omega_0, \qquad \omega_1'=(a-1)\omega_0+\omega_1, \qquad \omega_i'=\omega_i \quad (i=2, 3, \cdots, n).$$

We next consider the effect produced by moving x_1 positively around 1. By expanding ω_1 in powers of x_1-1 we readily find that it is reproduced multiplied by bl_1. The path of integration for ω_0 is changed into a path consisting of a line from 0 to 1, then an infinitesimal circle about 1 in the positive direction, and finally a line from 1 to x_1. This gives for the new integral $\omega_0+l_1(b-1)\omega_1$.

The path of integration for each of the other integrals we imagine to be gradually deformed by being pushed away from the moving point x_1. The new path

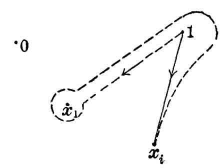

FIG. 2.

is indicated by the dotted line in Fig. 2, and the result of integrating along this is $\omega_i + b(l_1 - 1)\omega_1$. Thus we obtain the substitution

$$(\Sigma_1) \qquad \begin{aligned} &\omega_0' = \omega_0 + l_1(b-1)\omega_1, \qquad \omega_1' = bl_1\omega_1, \\ &\omega_i' = \omega_i + b(l_1 - 1)\omega_1 \qquad (i=2, \cdots, n). \end{aligned}$$

A similar mode of procedure will show that if $x_i(i > 1)$ move positively about the point 1 and then return to its original position, the integrals undergo the transformation

$$(\Sigma_i) \qquad \begin{aligned} &\omega_0' = \omega_0, \qquad \omega_h' = \omega_h + (l_i - 1)\omega_i \qquad (h=1, 2, \cdots, i-1), \\ &\omega_i' = bl_i\omega_i, \qquad \omega_k' = \omega_k + b(l_i - 1)\omega_i \qquad (k=i+1, \cdots, n). \end{aligned}$$

We now consider the effect of moving x_1 along a closed path encircling all the finite branch points, the direction of rotation being positive. In each of the integrals $\omega_i(i = 2, \cdots, n)$ the integrand is reproduced multiplied by l_1 while the path of integration is unaltered, and accordingly

$$(1) \qquad \omega_i' = l_1\omega_i.$$

For the integrals ω_0, ω_1 one of the limits of integration is variable. We therefore proceed as follows. Let a path be described from ∞ to x_n, then positively around all the finite branch points returning to x_n, then from x_n to ∞. The integral along such a path is zero. This gives the relation,

$$(2) \quad -\omega_n + b(\omega_1 - \omega_0) + ab\omega_0 + \sum_{h=1}^{n-1}\{n+1, h\}(\omega_{h+1} - \omega_h) + (L-1)I = 0,$$

in which for brevity we introduce the notation

$$l_0=a, \quad l_{n+1}=b, \quad \{h, k\}=l_h l_{h+1}\cdots l_k, \quad \{k, h\}=l_k l_{k+1}\cdots l_n l_{n+1} l_0 l_1 \cdots l_h (k>h),$$

$$L = abl_1\cdots l_n, \qquad I = \int_{x_n}^{\infty} U du.$$

After x_1 has traversed the path described above, equation (2) is changed into he same relation in ω_i' and I'. In this replace $\omega_2, \cdots, \omega_n$ by means of (1) and I' by the expression

$$I + (1 - l_1)\omega_n + b(1 - a)(1 - l_1)\omega_0 + b(l_1 - 1)\omega_1,$$

this being the result of integrating along the transformed path indicated in Fig. 3. Then replace I by means of (2) and we obtain the relation

$$(3)\qquad \begin{aligned} b(a-1)\omega_0' + b(1 - al_1)\omega_1' &= b(a-1)(l_1 + L - l_1 L)\omega_0 \\ &+ b[l_1(1-a) + L(1-l_1)]\omega_1 + (1-l_1)\sum_{h=2}^{n}(1-l_h)\{n+1, h-1\}\omega_h. \end{aligned}$$

Moreover, since the integral from 0 to 1 is reproduced multiplied by l_1, we have

$$(4)\qquad \omega_0' - \omega_1' = l_1(\omega_0 - \omega_1).$$

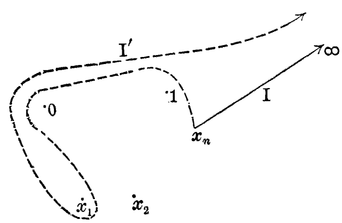

FIG. 3.

By combining (3) and (4) we obtain the formulas for ω_0', ω_1' and thus we deduce the required transformation:

$$(T_1)\qquad \begin{aligned} \omega_0' &= [1+(a-1)\{2, n+1\}]l_1\omega_0 + [\{2, n+1\} - 1]l_1\omega_1 + \sum_{h=2}^{n}(1-l_h)\{1, h-1\}\omega_h, \\ \omega_1' &= \{1, n+1\}[(a-1)\omega_0 + \omega_1] + \sum_{h=2}^{n}(1-l_h)\{1, h-1\}\omega_h, \qquad \omega_i' = l_1\omega_i \\ &\qquad\qquad (i = 2, 3, \cdots, n). \end{aligned}$$

By moving $x_i\,(i > 1)$ positively around all the branch points we obtain in like manner the transformations

$$(T_i)\qquad \begin{aligned} \omega_j' &= l_i\omega_j \qquad\qquad (j = 0, 1, \cdots, i-1, i+1, \cdots, n), \\ \omega_i' &= (a-1)\{i, n+1\}\omega_0 + (1 - al_1)\{i, n+1\}\omega_1 + \{i, i-1\}\omega_i \\ &\quad + \sum_{h=2,\ldots,i-1,i+1,\ldots,n}(1-l_h)\{i, h-1\}\omega_h. \end{aligned}$$

If x_1 describes a closed path circling positively around x_i (excluding all other branch points), we obtain the transformation

$$(\Sigma_{1i})\quad \begin{aligned} &\omega_0'=\omega_0+l_1(l_i-1)\omega_1+l_1(1-l_i)\omega_i, \qquad \omega_1'=[1+l_1(l_i-1)]\omega_1+l_1(1-l_i)\omega_i,\\ &\omega_h'=(1-l_1)(1-l_i)\omega_1+\omega_h-(1-l_1)(1-l_i)\omega_i \qquad (h=2,\cdots,i-1),\\ &\omega_i'=(1-l_1)\omega_1+l_1\omega_i, \qquad \omega_k'=\omega_k \qquad (k=i+1,\cdots,n).\end{aligned}$$

Finally when x_i describes a closed path about x_k we have

$$(\Sigma_{ik})\quad \begin{aligned} &\omega_h'=\omega_h \qquad (h=0,1,\cdots,i-1,k+1,k+2,\cdots,n),\\ &\omega_i'=[1+l_i(l_k-1)]\omega_i+l_i(1-l_k)\omega_k,\\ &\omega_j'=(1-l_i)(1-l_k)\omega_i+\omega_j-(1-l_i)(1-l_k)\omega_k \qquad (j=i+1,\cdots,k-1),\\ &\omega_k'=(1-l_i)\omega_i+l_i\omega_k.\end{aligned}$$

The above transformations evidently form a complete set of generators of the given group.

§ 2. *The invariant Hermitian form.*

Our next problem is to deduce the Hermitian form which is invariant for this group. Assuming the form to be

$$(5)\qquad \sum_{j,k=0}^{n} c_{jk}\omega_j\bar{\omega}_k \qquad (c_{jk}=\bar{c}_{kj}),$$

we first apply the transformation $T_i\ (i>1)$ and equate the coefficients of $\omega_k\bar{\omega}_i$ on both sides of the identity

$$\sum c_{jk}\omega_j'\bar{\omega}_k'=\sum c_{jk}\omega_j\bar{\omega}_k$$

after substituting in the left member for ω_j', $\bar{\omega}_k'$ the expressions given in (T_i). This leads to the formulas

$$(6)\qquad \begin{gathered} c_{0i}=c_{ii}\frac{(1-a)\{i,n+1\}}{l_i-L}, \qquad c_{1i}=c_{ii}\frac{(al_1-1)\{i,n+1\}}{l_i-L},\\ c_{ki}=c_{ii}\frac{(l_k-1)\{i,k-1\}}{l_i-L}.\end{gathered}$$

The coefficients of other corresponding terms are identical. Taking $k=2$ in the last of these formulas and replacing i by k we obtain

$$c_{2k}=c_{kk}\frac{(l_2-1)\{k,1\}}{l_k-L}.$$

Since $c_{2k}=\bar{c}_{k2}$ we deduce from this last equation

$$c_{k2}=c_{kk}\frac{(\bar{l}_2-1)\overline{\{k,1\}}}{\bar{l}_k-\bar{L}}=c_{kk}\frac{(1-\bar{l}_2)\{2,k\}}{(l_k-L)}.$$

Equate this to the result of putting $i=2$ in the last formula of (6) and we have

$$c_{kk}=c_{22}\frac{(1-\bar{l}_k)(l_k-L)}{(1-\bar{l}_2)(l_2-L)}. \tag{7}$$

By applying $\sum_0$ to (5) we obtain the additional relation

$$c_{01}=c_{11}\frac{a-1}{1-al_1}. \tag{8}$$

Also, by using $\sum_i$ we find the new condition

$$b(1-l_i)\sum_{h=1}^{i-1}c_{0h}+(1-bl_i)c_{0i}+(1-l_i)\sum_{k=i+1}^{n}c_{0k}=0.$$

In this we substitute c_{01} from (8), c_{0h}, c_{0k} from (6), and then c_{hh}, c_{kk} from (7). After using the identities

$$\sum_{h=2}^{i-1}(1-\bar{l}_h)\{h,\,n+1\}=\{2,\,n+1\}-\{i,\,n+1\},$$

$$\sum_{k=i+1}^{n}(1-\bar{l}_k)\{k,\,n+1\}=\{i+1,\,n+1\}-b,$$

we obtain the relation

$$c_{22}=c_{11}\frac{(1-\bar{l}_2)(l_2-L)}{(1-\bar{a}\bar{l}_1)\,(al_1-L)}.$$

By applying $\sum_1$ to the Hermitian form we find the additional relation

$$c_{00}(1-b)+c_{01}(1-bl_1)+\sum_{k=2}^{n}c_{0k}(1-l_1)=0,$$

from which we deduce, with the aid of the preceding results,

$$c_{11}=c_{00}\frac{(1-\bar{a}\bar{l}_1)(al_1-L)}{(1-\bar{a})(a-L)},$$

and hence

$$c_{kk}=c_{00}\frac{(1-\bar{l}_k)(l_k-L)}{(1-\bar{a})(a-L)},\qquad c_{01}=c_{00}\frac{\bar{l}_1L-a}{a-L},$$

$$c_{0i}=c_{00}\frac{\{i,\,0\}(\bar{l}_i-1)}{a-L},\qquad c_{1i}=c_{00}\frac{\{i,\,0\}(1-al_1)(1-\bar{l}_i)}{(1-a)(a-L)},$$

$$c_{ki}=c_{00}\frac{\{i,\,k\}(1-\bar{l}_i)(1-\bar{l}_k)}{(1-\bar{a})(a-L)}\qquad (k\neq i,\ i>1).$$

The Hermitian form is now completely determined, and it can be verified that all the remaining conditions are satisfied in order that H may be invariant for the generating substitutions of the group. I have carried out this verification, but the details are too long to reproduce here.

For certain cases in three homogeneous variables LE VAVASSEUR has calculated the numerical value of the determinant $|c_{ik}|$ of H and also of its minors. It is easy to evaluate this determinant for the general case. Every element contains the factor $C = c_{00}/(1-\bar{a})(a-L)$. We remove, then, the factor C^{n+1} and divide the elements of the first column by $1-\bar{a}$, those of the third by $1-\bar{l}_2$, those of the fourth by $1-\bar{l}_3, \cdots$, those of the $(n+1)$th by $1-\bar{l}_n$, and treat the resulting determinant as follows. Multiply the nth column by $-\bar{l}_{n-1}$ and add it to the last. Then multiply the $(n-1)$th column by $-\bar{l}_{n-2}$ and add to the nth, and so on; finally, multiply the first column by $\bar{a}\bar{l}_1$ and add to the third, and multiply the first column by $(1-\bar{a}\bar{l}_1)$ and add to the second. In the determinant thus formed, add the last row to the preceding, then the nth row to the $(n-1)$th, and so on; finally, add the third row to the second. Except for the first row and column, the elements outside the main diagonal of the resulting determinant are all zero and the evaluation is immediately effected. The result is $-L(1-\bar{b})(1-\bar{l}_1)(1-L)^n$. Combining this with the factors previously removed, we have

$$|c_{ik}| = -(a-1)(b-1)(l_1-1)(l_2-1)\cdots(l_n-1)(1-L)^n\left[\frac{c_{00}}{(1-\bar{a})(a-L)}\right]^n.$$

The determinant $|c_{ik}|$ *vanishes of rank* 1 *when* $L=1$. For it is easy to verify that every minor of order 2 vanishes identically when it contains no element from the main diagonal, while if it does contain such an element, the minor has $L-1$ as a factor. From this it follows that *in general H may be written* $A_0 y_0 \bar{y}_0 + (L-1)H_1$, *where H_1 is an Hermitian form in n variables* $y_1, y_2, \cdots, y_n$.

In the case of three variables $\omega_0, \omega_1, \omega_2$ we are able by means of this theorem to reduce H at once to a simple canonical form. Giving to c_{00} the value $(1-\bar{a})(a-L)(1-al_1)(\bar{a}\bar{l}_1-\bar{L})$ and introducing for brevity

$$\eta = (1-al_1)(1-bl_2)\omega_1 - (1-l_2)(1-al_1)\omega_2 - (1-a)(1-bl_2)\omega_0,$$

we obtain

$$H = \eta\bar{\eta} + (L-1)[(1-\bar{a})(1-\bar{b}\bar{l}_2)(1-\bar{l}_1)\omega_0\bar{\omega}_0 + (1-\bar{b})(1-\bar{l}_2)(1-\bar{a}\bar{l}_1)\omega_2\bar{\omega}_2].$$

§ 3. *Conditions for uniform inversion.*

We now introduce as non-homogeneous variables ξ_i the ratios of the hypergeometric integrals, $\xi_i = \omega_i/\omega_0$, and proceed to consider their developments in the vicinity of singular values of the variables x_i. For the sake of brevity the notation is made symmetrical by replacing the points $u=0, 1$ by x_0, x_{n+1} respectively and by using the integrals

$$\eta_i = \int_{x_0}^{x_i} U du$$

in place of the ω_i of which they are linear functions. Take now the singular point $x_h = x_k (h \neq k, h, k = 0, 1, \cdots, n+1)$, $x_i = a_i$ (the a_i being arbitrarily chosen but different from any of the branch points, $i \neq h, k$). Then all of the hypergeometric integrals are regular in the vicinity of such a point with the exception of

$$\eta_k - \eta_h = \int_{x_h}^{x_k} U du.$$

By substituting $x_k = x_h$, $u - x_h = v$, expanding U in ascending powers of v, and integrating between the limits 0 and x we find that $\eta_k - \eta_h$ has the form

$$\eta_k - \eta_h = x^{\lambda_h + \lambda_k - 1} R, \tag{9}$$

where R is regular in the vicinity of the given point.

We next examine the singularity $x_i = \infty$. By substituting $x_i = 1/x$, $u = 1/v$ it is readily seen that all of the integrals are expressible in the form $x^{1-\lambda_i} Q$, in which Q is regular, with the exception of the integral

$$\int_{x_i}^{\infty} U du$$

which has the form $x^{\mu} P$, in which P is regular and μ is

$$1 + \sum_{j=0}^{n+1} (1 - \lambda_j).$$

The quotients of these integrals are accordingly all regular with the exception of one which has the form

$$x^{\mu + \lambda_i - 1} \cdot S, \tag{10}$$

S being a regular function.

Replacing x_0, x_{n+1} by $0, 1$, we now consider the variables x_i as functions of the $\xi_i (i = 1, 2, \cdots, n)$. It is at once evident from what precedes that they are automorphic functions, being absolutely invariant for the group of linear transformations already determined. They are not in general uniform functions. It is interesting to determine under what conditions they are uniform. Since the expressions (9) and (10) are linear fractional functions of the ξ_i, it is evidently necessary that the exponents of the x, namely,

$$\lambda_h + \lambda_k - 1, \qquad n + 2 - (\lambda_0 + \lambda_1 + \cdots + \lambda_{i-1} + \lambda_{i+1} + \cdots + \lambda_{n+1}),$$

be each the reciprocal of an integer. There are

$$\tfrac{1}{2}(n+2)(n+1) + n + 2 = \tfrac{1}{2}(n+2)(n+3)$$

such conditions, except in the case $n = 1$ when the number is 3. The number of different solutions of the problem of determining the exponents λ_i so as to satisfy these conditions is infinite for $n = 1$ and finite for $n = 2$.* As the

*LE VAVASSEUR, Comptes Rendus, vol. 115 (1892), p. 1006.

number of conditions increases rapidly with increasing n, it may be presumed that the number of solutions is finite in each case excepting $n = 1$.

§ 4. *Case of equal exponents.*

When any two or more exponents in the hypergeometric integral are equal, the group may be extended. For this purpose let the branch points $0, 1$ be replaced by x_0, x_{n+1} as in the preceding section. Take the case $\lambda_0 = \lambda_1$ and consider first a variation of the points x_0 and x_1 by which they become permuted, the paths described by these two points forming a closed region not containing any of the other branch points. It will be assumed in this and the following cases that the paths are described positively with reference to the region they enclose. The following method, which might also have been used in § 2, is convenient for the case in hand. Take an arbitrary point u_0 of the u-plane and denote by I_0, I_1 the result of integrating $\int U\,du$ along a convenient path from u_0 to x_0, x_1 respectively. We will denote these paths by p_0, p_1. When x_0 and

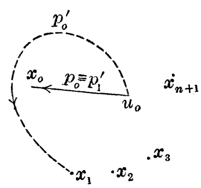

FIG. 4.

x_1 are interchanged these paths will be changed continuously into new paths p_0', p_1' (Fig. 4). Then we have

$$I_0' = I_0 + l_0\omega_0, \qquad I_1' = I_0, \qquad I_k' = I_k \quad (k = 2, \cdots, n+1),$$

and hence

$$\omega_0' = I_1' - I_0' = l_0\omega_0, \quad \omega_1' = I_1' - I_{n+1}' = \omega_1 - \omega_0, \quad \omega_j' = \omega_j \quad (j = 2, \cdots, n).$$

In general, if $\lambda_i = \lambda_k$, we deduce in like manner the following results in which the symbol (ik) is used to indicate the substitution which results from the permutation of the branch points x_i, x_k:

$$(0i) \quad \begin{aligned} &\omega_0' = \omega_1 - \omega_i, \quad \omega_h' = (1 - l_0)(\omega_1 - \omega_0 - \omega_i) + \omega_h \quad (h = 1, 2, \cdots, i-1;\ i = 2, \cdots, n), \\ &\omega_i' = \omega_1 - \omega_0, \quad \omega_j' = \omega_j \qquad\qquad (j = i+1, \cdots, n); \end{aligned}$$

$$(0, n+1) \qquad \omega_0' = \omega_1, \qquad \omega_i' = \omega_i + l_0(\omega_0 - \omega_1) \qquad (i = 1, \cdots, n);$$

$$(1i) \quad \begin{aligned} &\omega_0' = \omega_0 + l_1(\omega_i - \omega_1), \qquad \omega_1' = (1 - l_1)\omega_1 + l_1\omega_i, \\ &\omega_h' = (1 - l_1)(\omega_1 - \omega_i) + \omega_h \qquad (h = 2, 3, \cdots, i-1), \\ &\omega_i' = \omega_1, \qquad \omega_j' = \omega_j \qquad (j = i+1, \cdots, n); \end{aligned}$$

$$(1, n+1) \quad \omega_0' = \omega_0 - l_1 \omega_1, \qquad \omega_1' = - l_1 \omega_1, \qquad \omega_i' = \omega_i - l_1 \omega_1 \quad (i = 2, \cdots, n);$$

$$(ik) \quad \begin{aligned} &\omega_h' = \omega_h, \qquad \omega_k' = \omega_i \quad (h = 1, 2, \cdots, i-1, k+1, \cdots, n), \\ &\left.\begin{aligned} \omega_i' &= (1 - l_i)\omega_i + l_i \omega_k, \\ \omega_j' &= (1 - l_i)(\omega_i - \omega_k) + \omega_j \end{aligned}\right\} \quad (j = i+1, \cdots, k-1), \\ &(1 < i < k < n+1). \end{aligned}$$

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